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1980 J. Phys. A: Math. Gen. 13 1685

(http://iopscience.iop.org/0305-4470/13/5/026)

J. Phys. A: Math. Gen. **13** (1980) 1685–1699. Printed in Great Britain

# Non-adiabatic parametric excitation of oscillator-type systems

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Received 18 June 1979, in final form 8 October 1979

**Abstract.** A harmonic quantum oscillator and the motion of charged particles in a magnetic field have been considered, the dependence on time of the magnetic field and the oscillator frequency being such that they permit us to keep an exact record of non-adiabaticity. For these systems, exact solutions, motion integrals, coherent states and Green functions have been constructed and proper amplitudes and transition probabilities calculated. One of the possible practical applications of the results obtained has been pointed out, namely their use for the interpretation of experiments with electronic vibrational transitions in molecules.

## 1. Introduction

A non-stationary harmonic oscillator with an arbitrarily time-dependent vibration frequency and the motion of a charged particle in a time-dependent magnetic field are dealt with in a number of papers. For example, Husimi (1953) has obtained a solution of Schrodinger's equation, Green's function and the matrix element of an evolution operator in the expanded form for a one-dimensional quantum oscillator with a varying frequency.

Lewis and Riesenfeld (1969) have developed a theory of time-dependent invariants for non-stationary quantum systems and applied it to a one-dimensional harmonic oscillator with a varying frequency and to the motion of a charged particle in a time-dependent electromagnetic field.

The authors of another paper (Malkin *et al* 1970) have studied the quantum-mechanical problems of an $N$-dimensional oscillator with an arbitrary dependence of frequencies on time and those of a non-relativistic charged particle in an axially symmetrical non-stationary electromagnetic field. They have also shown that a quantum oscillator problem involves the solution of an appropriate equation for a classical oscillator.

Only approximate solutions are usually possible for problems associated with oscillator-type non-stationary systems. So, for instance, a harmonic oscillator having a frequency varying slowly with time was treated by Kulsrud (1957) and Kruskal (1961) in connection with the working out of the theory of adiabatic invariants. Elsewhere (Zeldovich and Starobinsky 1971) an asymptotic solution to a classical oscillator equation for the case of large impulses was obtained in a study concerning the problems of particle production and vacuum polarisation in an anisotropic gravitational field.

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0305-4470/80/051685+15$01.50  

At present, several papers are known to deal with a non-stationary harmonic oscillator involving a specific dependence of frequency on time.

An equation for a classical oscillator with a varying frequency with the forms

$$w^2(t) = C_1(1 + C_2t + C_3t^2) \qquad \text{and} \qquad w^2(t) = C + \tfrac{1}{4}t^2$$

is solved in an earlier paper (Grib *et al* 1974), in connection with a problem involving the production of particles from a vacuum in a non-stationary isotropic universe. It is known (Malkin *et al* 1970) that exact solutions for a non-stationary oscillator can also be obtained at an instantaneous frequency variation from one constant value to another for a frequency having the form $w^2(t) = C + (C_1/\cosh^2 C_2t)$ and for a varying frequency complying with the Eckart potential.

The present paper is aimed at solving a non-stationary problem both for a one-dimensional harmonic quantum oscillator and for a charged particle moving in a magnetic field, the time dependence of the frequency (in the case of a magnetic field this frequency is proportional to the cyclotron frequency) being

$$\begin{aligned} w &= a + b && \text{for } t \leqslant 0 \\ w(t) &= a + b\exp(-\varkappa t) && \text{for } t \geqslant 0, \end{aligned} \tag{1}$$

i.e. it is aimed at determining all the linear system motion integrals, constructing the coherent states, and using them to calculate the Green function, the amplitude and the probability of transitions between the energy levels. In formulae (1) the parameters $a$, $b$, and $\varkappa$ are arbitrary non-negative constants.

In relation to the frequency, the dependence (1) under consideration in this paper corresponds to one more case exactly solved in explicit form, as compared to Malkin *et al* (1970) and Grib *et al* (1974). It should be noted that in non-stationary problems the above dependence (1) for $w(t)$ has not been considered up to now. Besides, the interest displayed in the problem under consideration is due also to the fact that the results obtained may be used for practical purposes, for example, to interpret experimental data on electronic vibrational (vibronic) transitions in molecules.

Actually, in previous reports dealing with analytical methods of studying vibronic spectra of polyatomic molecules, as in Doktorov *et al* (1976), an adiabatic approximation was used, i.e. it was supposed that an electronic transition occurred instantly when the nuclei positions were not yet changed.

Two other papers (Doktorov *et al* 1975, Malkin and Man'ko, 1975) pointed to the possibility of studying the time responses of vibronic transitions in polyatomic molecules within the framework of the Born–Oppenheimer approximation, as well as the harmonic approximation for nuclei vibrations. In this case, the electronic motion may be regarded to be not instantaneous with respect to the nuclei motion, but of a sufficiently long duration, i.e. non-adiabatic. Such a phenomenon may be important, if one considers the orbit-to-orbit transition of an electron which is not a light one but a heavier negative muon in a polyatomic mesomolecule. Of some interest in this connection are the models adequate to various time processes. One such model may be represented by a relaxation model in which the duration of the transition process (the transition of an electron or a muon) is given by the dependence (1) where $\varkappa^{-1}$ is the characteristic time of the transition. The present paper deals with an investigation of such a model, a one-dimensional harmonic oscillator and the motion of a charged particle in a magnetic field being used as examples.

In order to provide a complete understanding, formulae relating to a quantum oscillator with an arbitrary time-dependence of frequency are presented in this paper, in conformity with an earlier paper (Malkin *et al* 1970).

## 2. Motion integrals for a non-stationary quantum oscillator

Let us consider a quantum system described by the Hamiltonian

$$\hat{\mathscr{H}} = \hat{p}^2/2m + \tfrac{1}{2}mw^2(t)\hat{x}^2 \tag{2}$$

where $x$ is a usual canonical coordinate, $p$ is its conjugate momentum, and $m$ is the mass.

The wave equation has the form

$$(\mathrm{i}\hbar\partial/\partial t - \hat{p}^2/2m - \tfrac{1}{2}mw^2(t)\hat{x}^2)\psi = 0. \tag{3}$$

According to the question concerning motion integrals, discussed in the above report (Malkin *et al* 1970), $2N$ independent non-Hermitian invariants must exist for systems with $N$ degrees of freedom. One can make sure immediately that the operator

$$\hat{A}(t) = \frac{\mathrm{i}}{(2\hbar)^{1/2}}\left(\frac{\hat{p}}{\sqrt{m}}\epsilon(t) - \sqrt{m}\hat{x}\dot{\epsilon}(t)\right), \tag{4}$$

where the function $\epsilon(t)$ is a definite solution to the equation of the classical harmonic oscillator

$$\ddot{\epsilon} + w^2(t)\epsilon = 0, \tag{5}$$

commutes with the operator $(\mathrm{i}\hbar\partial/\partial t - \hat{\mathscr{H}})$ and is thus a motion integral. The following commutation relationship holds:

$$[\hat{A}, \hat{A}^+] = 1. \tag{6}$$

Formulae (5) and (6) give the following equality,

$$\dot{\epsilon}\epsilon^* - \dot{\epsilon}^*\epsilon = 2\mathrm{i}, \tag{7}$$

which is valid for any moment of time $t$.

Let us present the motion integrals $\hat{A}$ and $\hat{A}^+$ in the form (Malkin and Man'ko 1975)

$$\begin{aligned}\hat{A} &= (\hat{X}_0 + \mathrm{i}\hat{P}_0)/(2\hbar)^{1/2},\\ \hat{A}^+ &= (\hat{X}_0 - \mathrm{i}\hat{P}_0)/(2\hbar)^{1/2}.\end{aligned} \tag{8}$$

Comparing these formulae with (4), we obtain the following expressions for the invariant $\hat{X}_0$ referred to as the operator of the initial coordinate, and the invariant $\hat{P}_0$ called the operator of the initial impulse:

$$\begin{aligned}\hat{X}_0 &= \tfrac{1}{2}\mathrm{i}[\sqrt{m}\hat{x}(\dot{\epsilon}^* - \dot{\epsilon}) + (\hat{p}/\sqrt{m})(\epsilon - \epsilon^*)],\\ \hat{P}_0 &= \tfrac{1}{2}[(\hat{p}/\sqrt{m})(\epsilon + \epsilon^*) - \sqrt{m}\hat{x}(\dot{\epsilon}^* + \dot{\epsilon})].\end{aligned} \tag{9}$$

## 3. Invariants for a charged particle in a non-stationary magnetic field

Let us consider a frequency with a mass $m$ and a charge $e$, moving in a classical magnetic field with the potential

$$\boldsymbol{A}(\boldsymbol{r}, t) = \tfrac{1}{2}[\boldsymbol{H}(t), \boldsymbol{r}], \tag{10}$$

where $\boldsymbol{r}$ is the position vector, and $\boldsymbol{H}(t)$ is a uniform magnetic field. We choose the $z$-axis along the magnetic field $\boldsymbol{H}$, and then $A_z = 0$. Then the motion along the $z$-axis is trivial and can be neglected. We shall consider only the motion in the $xy$ plane. The Hamiltonian for such a system is

$$\hat{\mathscr{H}} = \frac{1}{2m}\left[\left(\hat{p}_x - \frac{e}{c}A_x\right)^2 + \left(\hat{p}_y - \frac{e}{c}A_y\right)^2\right]. \tag{11}$$

We omit the spin-dependent term $\mu\sigma_z H$ inasmuch as the subsequent description is independent of it.

By direct calculation one can make sure that the following operators are invariants:

$$\hat{A}(t) = f(t)[\epsilon(t)(\hat{p}_x + \mathrm{i}\hat{p}_y) - \mathrm{i}m\dot{\epsilon}(t)(y - \mathrm{i}x)]/(2\hbar m)^{1/2}, \tag{12}$$

$$\hat{B}(t) = f(t)[\epsilon(t)(\hat{p}_y + \mathrm{i}\hat{p}_x) - \mathrm{i}m\dot{\epsilon}(t)(x - \mathrm{i}y)]/(2\hbar m)^{1/2}, \tag{13}$$

where $\epsilon(t)$ is a certain specific solution to equation (5),

$$f(t) = \frac{1}{\sqrt{2}}\exp\left(\mathrm{i}\int_0^t w(\tau)\,\mathrm{d}\tau\right), \tag{14}$$

and in this case $w(t)$ should be understood not as the oscillator frequency but as

$$w(t) = eH(t)/2mc. \tag{15}$$

Operators (12) and (13) commute with the operator $(\mathrm{i}\hbar\partial/\partial t - \hat{\mathscr{H}})$, because $\dot{\hat{A}} = \dot{\hat{B}} = 0$.

The Vronskian of the system consisting of equation (5) and an equation complex conjugate to equation (5) must be equal to zero.

A straightforward calculation shows that the value $(\dot{\epsilon}\epsilon^* - \epsilon\dot{\epsilon}^*)$ is then independent of time, i.e. it is constant. This constant can be determined on the basis of the initial conditions.

Let us suppose that for $t \to \infty$ and $t \leqslant 0$ the magnetic field assumes constant values of $H_f$ and $H_i$ so that $w$ is equal to $w_f$ and $w_i$, respectively. Then for $t \leqslant 0$ we choose as a solution to equation (5)

$$\epsilon_i = w_i^{-1/2}\exp(\mathrm{i}w_i t), \tag{16}$$

which gives a value of the constant equal to 2i. Thus, in the case of a magnetic field too, formula (7) is valid, and on this basis one can obtain the following commutation relationships:

$$[\hat{A}, \hat{A}^+] = [\hat{B}, \hat{B}^+] = 1, \qquad [\hat{A}, \hat{B}] = [\hat{A}, \hat{B}^+] = 0. \tag{17}$$

Four more invariants, namely $\hat{X}_0$, $\hat{Y}_0$, $\hat{P}_{0x}$ and $\hat{P}_{0y}$, may be introduced, which will be necessary later, if one presents $\hat{A}$, $\hat{B}$, $\hat{A}^+$, and $\hat{B}^+$ as

$$\hat{A} = (\hat{X}_0 + \mathrm{i}\hat{P}_{0x})/(2\hbar)^{1/2}, \qquad B = (\hat{Y}_0 + \mathrm{i}\hat{P}_{0y})/(2\hbar)^{1/2},$$

$$\hat{A}^+ = (\hat{X}_0 - \mathrm{i}\hat{P}_{0x})/(2\hbar)^{1/2}, \qquad B^+ = (\hat{Y}_0 - \mathrm{i}\hat{P}_{0y})/(2\hbar)^{1/2},$$

and uses expressions (12) and (13). To take an example, we shall obtain

$$\hat{X}_0 = \tfrac{1}{2}m^{-1/2}[(\epsilon f + \epsilon^* f^*)\hat{p}_x + (\epsilon f - \epsilon^* f^*)\mathrm{i}\hat{p}_y - \mathrm{i}(\dot{\epsilon} f - \dot{\epsilon}^* f^*)my - (\dot{\epsilon} f + \dot{\epsilon}^* f^*)mx]. \tag{18}$$

The expression for $\mathrm{i}\hat{P}_{0x}$ is derived from (18) at $f \to -f$, whereas $\hat{Y}_0$ and $\hat{P}_{0y}$ are obtained from $\hat{X}_0$ and $\hat{P}_{0x}$, respectively, at $x \leftrightarrow y$, $f \leftrightarrow f^*$, and $\hat{p}_x \leftrightarrow \hat{p}_y$.

Two more integrals of motion in a magnetic field have been presented elsewhere (Malkin *et al* 1970). These are the Hermitian operator $\hat{K} = \hat{A}^+\hat{A} + \frac{1}{2}$ and the $z$ component of the angular moment $\hat{L}_z = \hat{B}^+\hat{B} - \hat{A}^+\hat{A}$. But a quantum system with four degrees of freedom ($x$, $y$-plane motion) can have only four independent motion integrals. It should be noted that only four of the invariants enumerated above are independent motion integrals in accordance with the four degrees of freedom of the system under discussion.

## 4. Exact solution of equation for $\epsilon(t)$

From the foregoing it transpires that one should solve equation (5) to determine both the motion integrals of a quantum oscillator and the motion integrals in a magnetic field.

Let us solve equation (5) for the case of the dependence $w(t)$ specified by formula (1).

For $t \leqslant 0$ we assume

$$\epsilon(t) \equiv \epsilon_i = w_i^{-1/2} \exp(\mathrm{i}w_i t),$$
$$w_i = a + b, \qquad b \geqslant 0,\ a > 0;\ a, b < \infty. \tag{19}$$

In order to solve this equation at $t \geqslant 0$, we make the following substitution in equation (5):

$$x = x_0 \exp(-\varkappa t), \qquad x_0 = 2\mathrm{i}(w_i - w_f)\varkappa^{-1},$$
$$\epsilon(t) = x^{-1/2} y(x), \qquad w_f = a, \tag{20}$$

and give the notation

$$l = -k = \mathrm{i}w_f \varkappa^{-1}. \tag{21}$$

The equation (5) reduces to the Whittaker equation

$$4x^2 y = (x^2 - 4kx + 4l^2 - 1)y. \tag{22}$$

Its solution is

$$\epsilon(t) = x^{-1/2}[AM_{k,-k}(x) + BW_{k,-k}(x)] \tag{23}$$

where $A$ and $B$ are constants, and $M$ and $W$ are Whittaker's functions (Gradshteyn and Ryzhik 1971).

Taking into account the expression of Whittaker's functions in terms of a degenerate hypergeometric function (Gradshteyn and Ryzhik 1971)

$$\phi(a, c, z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \tag{24}$$

and the fact that in the case under discussion $2l = 2ia\varkappa^{-1}$ is not an integer, we transform equation (23) to the following form:

$$\epsilon(t) = w_f^{-1/2}[\xi(t)\exp(\mathrm{i}w_f t) - \eta(t)\exp(-\mathrm{i}w_f t)]. \tag{25}$$

Here, the time-dependent complex functions $\xi$ and $\eta$ are

$$\xi(t) = \xi_0\left[\exp\left(\frac{x_0 - x}{2}\right)\right]\phi_2/\phi_2^0, \qquad \eta(t) = \eta_0\left[\exp\left(\frac{x_0 - x}{2}\right)\right]\phi_1/\phi_1^0. \tag{26}$$

The index '0' means that the appropriate values have been taken at the point $t = 0$. For the hypergeometric function $\phi_1$, $a = -2k + \frac{1}{2}$, $c = 1 - 2k$, $z = x$, whereas for $\phi_2$, $a = \frac{1}{2}$, $c = 1 + 2k$, $z = x$.

The constants $\xi_0$ and $\eta_0$ will be determined from the continuity conditions at the point $t = 0$ of the functions $\epsilon_1$ and $\epsilon_2$ themselves as well as their derivatives.

When calculating $\xi_0$ and $\eta_0$, we start from the equality

$$\dot{\xi}\alpha = \dot{\eta}\beta = 0, \tag{27}$$

where $\alpha$ is equal to the coefficient at $\xi$, and $\beta$ is equal to the coefficient at $\eta$ in formula (25). It can be shown that equality (27) will occur only for a certain relation between $\alpha$ and $\beta$, namely the condition $\ddot{\alpha}\beta = \alpha\ddot{\beta}$. It is easy to make sure that the coefficients at $\xi$ and $\eta$ from equation (25) satisfy this condition. With due regard for what we said about $\dot{\epsilon}$, we obtain the following expression:

$$\dot{\epsilon}(t) = \mathrm{i}\sqrt{w_f}[\xi(t)\exp(\mathrm{i}w_f t) + \eta(t)\exp(-\mathrm{i}w_f t)]. \tag{28}$$

Finally we have

$$\xi_0 = (w_i + w_f)/2(w_i w_f)^{1/2},$$
$$\eta_0 = (w_i - w_f)/2(w_i w_f)^{1/2}. \tag{29}$$

Substituting equations (25) and (28) into relation (7), which holds for the two systems considered by us, we obtain the equality

$$|\xi(t)|^2 - |\eta(t)|^2 = 1, \tag{30}$$

which will be used by us in subsequent calculations.

On the basis of formula (25) one can consider a number of specific cases.

(1) $b = 0$. As might be expected, in this case we obtain

$$\epsilon(t) = w_f^{-1/2}\exp(\mathrm{i}w_f t) \equiv \epsilon_f. \tag{31}$$

(2) $\quad \varkappa \to 0 \qquad \epsilon(t) = \epsilon_i. \qquad (32)$

(3) $\quad \varkappa \to \infty \qquad \epsilon(t) = w_f^{-1/2}[\xi_0\exp(\mathrm{i}w_f t) - \eta_0\exp(-\mathrm{i}w_f t)]. \qquad (33)$

(4) $\quad t \to \infty \qquad \epsilon(t) \equiv \epsilon_\infty = w_f^{-1/2}[\xi_\infty\exp(\mathrm{i}w_f t) - \eta_\infty\exp(-\mathrm{i}w_f t)], \qquad (34)$

where the complex numbers $\xi_\infty$ and $\eta_\infty$ are determined from equation (26) as $t \to \infty$:

$$\xi_\infty = \xi_0(\phi_2^0)^{-1}\exp(x_0/2),$$
$$\eta_\infty = \eta_0(\phi_1^0)^{-1}\exp(x_0/2). \tag{35}$$

It should be noted that the knowledge of the exact expression of equation (25) makes it possible to construct proper coherent states and Green functions, and calculate the amplitudes and probabilities of transitions for the systems under discussion.

## 5. Coherent states and Green's function for an oscillator

Now we shall construct coherent states for an oscillator with a time-dependent frequency (1) as normalised eigenfunctions of the motion integrals (4):

$$\hat{A}|\alpha t\rangle = \alpha|\alpha, t\rangle \tag{36}$$

where $\alpha$ is an arbitrary complex number.

Then we have

$$|\alpha, t\rangle = \left(\frac{m}{\pi\hbar\epsilon^2}\right)^{1/4} \exp\left[\frac{\mathrm{i}\dot{\epsilon}}{2\epsilon}\left(\sqrt{\frac{m}{\hbar}}\,x - \frac{\mathrm{i}\sqrt{2}}{\dot{\epsilon}}\,\alpha\right)^2 - \frac{1}{2}\left(\frac{\dot{\epsilon}^*}{\dot{\epsilon}}\,\alpha^2 + |\alpha|^2\right)\right]. \tag{37}$$

The wavefunction $|\alpha, t\rangle$ obeys the Schrödinger equation with the Hamiltonian (2) and the normalisation condition. Coherent states form a complete system, but they are not orthogonal.

A coherent state is a generating function for the eigenfunctions of the Hermitian operator $\hat{A}^+\hat{A}$:

$$|\alpha, t\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n, t\rangle. \tag{38}$$

The function $|n, t\rangle$ obeys the condition

$$\hat{A}^+\hat{A}|n, t\rangle = n|n, t\rangle \tag{39}$$

where $n$ stands for non-negative integers.

The explicit form of eigenfunctions $|n, t\rangle$ can be obtained from the explicit form of coherent states $|\alpha, t\rangle$, if we use the generating function for the Hermite polynomials (Bateman and Erdelyi 1966):

$$\exp(-s^2 + 2sx) = \sum_{n=0}^{\infty} \frac{1}{n!}\, s^n H_n(x). \tag{40}$$

Expanding the coherent state $|\alpha, t\rangle$ in a series of the variable $\alpha$, we obtain

$$|n, t\rangle = (\epsilon^*/2\epsilon)^{n/2}(n!\epsilon)^{-1/2}(\pi\hbar/m)^{-1/4} \exp(\mathrm{i}\dot{\epsilon}mx^2/2\hbar\epsilon) H_n\left(\sqrt{\frac{m}{\hbar}}\,x/|\epsilon|\right). \tag{41}$$

Functions (41) are orthonormal.

As seen from equation (37), the coherent states of a one-dimensional oscillator with a time-dependent frequency are solutions of a Gaussian form, i.e. an exponential of a quadratic. Gaussian packets for a non-stationary one-dimensional oscillator were first constructed in an earlier study (Husimi 1953) where the Green function was also found for such a system. Using the coherent states, it is easy to obtain this Green function for the system under consideration with the aid of the completeness relation, by calculating the integral

$$\pi^{-1} \int \mathrm{d}^2\alpha\, |\alpha, x_2, t_2\rangle\langle\alpha, x_1, t_1|.$$

As a result we obtain the following simplified expression of the Green function, as compared to the expression of Husimi (1953) and Malkin *et al* (1970):

$$G(x_2, t_2, x_1, t_1) = (2\mathrm{i}\pi\hbar|\epsilon_1\epsilon_2|\sin\gamma/m)^{-1/2}\exp\left[-\frac{\mathrm{i}}{\sin\gamma}Q_1Q_2 + \frac{\mathrm{i}}{2}(Q_1^2+Q_2^2)\cot\gamma + \frac{\mathrm{i}}{4}\left(Q_2^2\frac{\mathrm{d}|\epsilon_2|^2}{\mathrm{d}t_2} - Q_1^2\frac{\mathrm{d}|\epsilon_1|^2}{\mathrm{d}t_1}\right)\right]. \tag{42}$$

Here,

$$Q_{1,2} = \sqrt{\frac{m}{\hbar}}\frac{x_{1,2}}{|\epsilon_{1,2}|}, \qquad \sin\gamma = \frac{z^2-1}{2\mathrm{i}z}, \qquad z = (\epsilon_1\epsilon_2^*/\epsilon_1^*\epsilon_2)^{1/2}$$

and $\mathrm{d}^2\alpha$ represents a differential of the form $\mathrm{d}^2\alpha = \mathrm{d}\,\mathrm{Re}\,\alpha\,\mathrm{d}\,\mathrm{Im}\,\alpha$. When calculating expression (42), one may either integrate directly, keeping in mind equation (37), or use formula (38) and the formula (Bateman and Erdelyi 1966):

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^n H_n(Q_1)H_n(Q_2) = \frac{1}{(1-z^2)^{1/2}}\exp\left(\frac{2Q_1Q_2z-(Q_1^2+Q_2^2)z^2}{1-z^2}\right). \tag{43}$$

In order to elucidate the physical meaning of the coherent states $|\alpha, t\rangle$ and the integrals of motion $\hat{A}(t)$, we shall consider their limiting expressions when the vibration frequencies $w(t)$ tend to constant values. For the sake of simplicity we suppose that $w(t) = w_i$ for $t \leqslant 0$ and $w(t) = w_f$ when $t \to \infty$, where $w_{i,f}$ are constants (see (19) and (20)). Then, within the limits of $t \to \pm\infty$, there are complete systems of functions, initial coherent states $|\alpha, i\rangle(\epsilon = \epsilon_i)$ and final coherent states $|\beta, i\rangle(\epsilon = \epsilon_f)$ as well as orthonormalised complete systems of functions which are eigenfunctions with specified vibration energies, initial states $|n, i\rangle$ and final states $|m, f\rangle$.

Between the initial and final states transitions take place, and one can calculate the amplitude of these transitions. As the result of a quantum transition, the system will pass from the initial state (in our case, for example $|i\rangle$ with $\epsilon = \epsilon_i$) to a state with $\epsilon = \epsilon_\infty$ (see formula (34)), which can be expanded in a complete system of final stationary functions $|f\rangle(\epsilon = \epsilon_f)$. The amplitude of the probability for the system, after the transition, to end up in one of the possible final states, for example in $|f_1\rangle$, is equal to the coefficient at $|f_1\rangle$ in the expansion obtained. The general expression for the transition amplitude relating the initial state $|i\rangle$ to the final state $|f\rangle$ is given by the matrix element

$$T_i^f = \langle f|t \to \infty\rangle \tag{44}$$

where $|t \to \infty\rangle(\epsilon = \epsilon_\infty)$ is the limit as $t \to \infty$ of the state $|t\rangle$ with the initial state $|i\rangle$ as its limiting value at negative time values $t \to -\infty$.

We have chosen such initial conditions for the solution of equation (5) in order to have a correct limiting value as $t \to -\infty$ for the coherent state $|\alpha, t\rangle$, i.e. $\epsilon = \epsilon_i$ (see (19)).

By substituting $\epsilon = \epsilon_i$ into (4) we obtain an expression for the initial integral of motion $\hat{A}_i$. Then it is obvious that the states $|\alpha, -\infty\rangle$ and $|n, -\infty\rangle$ will coincide with the initial states $|\alpha, i\rangle$ and $|n, i\rangle$, constructed by means of operators (integrals of motion for a stationary oscillator) $\hat{A}_i$ with the use of the same formulae which were used to construct the states $|\alpha, t\rangle$ and $|n, t\rangle$ by means of operators $\hat{A}(t)$. That is why the expressions for the initial states $|\alpha, i\rangle$ and $|n, i\rangle$ are given by formulae (37) and (41), respectively, if we substitute $w \to w_i$ and $\epsilon = \epsilon_i$ in them.

The final operator $\hat{A}_f$, pertaining to the constant frequency $w_f$, and the final states $|\gamma, f\rangle$ and $|m, f\rangle$ are obtained from formulae (4), (37) and (41) while substituting $w = w_f$,

$\epsilon = \epsilon_f$. By checking directly, one can make sure that the following expansion holds for operator (4):

$$\hat{A}(t) = \xi(t)\hat{A}_f + \eta(t)\hat{A}_f^+. \tag{45}$$

The operators $\hat{X}_0$ and $\hat{P}_0$ introduced in § 2 are actually initial-coordinate and initial-impulse operators, because they represent an invariant combination of $\hat{x}$, $\hat{p}$ and $t$ and, as may be seen from (9) and (19), the operators $\hat{X}_0$ and $\hat{P}_0$ reduce just to the usual operators of coordinates and impulses at $t = 0$.

From (36) and the condition of normalisation of the functions $|\alpha, t\rangle$ it follows that the mean value of $\hat{A}$ in the state $|\alpha, t\rangle(\langle\hat{A}\rangle)$ is equal to the complex number $\alpha$, and if one takes into account the expression of $\hat{A}$ in terms of $\hat{X}_0$ and $\hat{P}_0$ (equations (8)) then

$$\hat{x} = (mw_i)^{-1/2}\hat{X}_0(t=0), \qquad \hat{p} = (mw_i)^{1/2}\hat{P}_0(t=0) \tag{46}$$

where $\alpha_1 = \mathrm{Re}\,\alpha$, $\alpha_2 = \mathrm{Im}\,\alpha$. It is known that the motion on a phase plane $(\alpha_1, \alpha_2)$ can be presented as the motion of a classical oscillator with coordinate $x$ and impulse $p$, if one assumes $\alpha_1 = \sqrt{2M\Omega}x$ and $\alpha_2 = (2/M\Omega)^{1/2}p$ ($M$ is the mass, and $\Omega$ is the frequency of a classical oscillator). It follows that a quantum oscillator in the coherent state is extremely close to a classical oscillator in the sense that the mean values of the initial coordinate $\langle\hat{X}_0\rangle$ and the initial impulse $\langle\hat{P}_0\rangle$ are similar to the coordinates $x$ and $p$ of a classical oscillator. Then by analogy with the classical oscillator, the value of $|\alpha|$ defines the classical amplitude of vibrations of the quantum oscillator, whereas the phase $\varphi(\alpha)$ defines the classical phase of vibrations of the same oscillator (the specific expression for $\alpha$ is taken from (46)).

## 6. Probability of transitions for a non-stationary quantum oscillator

Now, let us direct our attention to the calculation of the amplitudes of transitions (44) for a non-stationary oscillator. From the very meaning of expression (44) it is clear that all the amplitudes we are going to consider are defined completely by the constants $\xi_\infty$ and $\eta_\infty$ (35).

The amplitudes of transitions between various states for a one-dimensional quantum oscillator were obtained by Husimi (1953). Let us derive formulae for the amplitudes of transitions between coherent states:

$$T_\alpha^\gamma = \xi_\infty^{-1/2} \exp\{\tfrac{1}{2}[\alpha^2\eta_\infty^*/\xi_\infty + \alpha\gamma^* 2/\xi_\infty - (\gamma^*)^2\eta_\infty/\xi_\infty - |\alpha|^2 - |\gamma|^2]\}. \tag{47}$$

Given the amplitude of a transition between coherent states, one can obtain the amplitudes of a transition from a coherent state to a state with a specified energy $|\alpha, t\rangle \to |n, t\rangle$, and vice versa, $|n, t\rangle \to |\alpha, t\rangle$, if the generating function of a Hermite polynomial (40) and formula (38) are used.

The amplitude of a transition $T_n^m$ can be calculated either by taking into account the fact that $T_\alpha^\gamma \exp[\frac{1}{2}(|\alpha|^2 + |\gamma|^2)]$ is a generating function for associated Legendre polynomials (Malkin *et al* 1970):

$$\exp(a^2 + b^2 + 2abx) = \sum_{m,n=0}^{\infty} a^m b^n (\mathrm{i}^k k!)^{-1}[2(1-x^2)^{1/2}]^{(m+n)/2} P_{(m+n)/2}^{|m-n|/2}\left(\frac{\mathrm{i}x}{(1-x^2)^{1/2}}\right),$$

$$k = \tfrac{1}{2}(m + n + |m-n|), \tag{48}$$

or by integrating directly, with due regard for the fact that $(m \geqslant n)$ (Bailey 1948)

$$\int_{-\infty}^{+\infty} \exp(-\alpha x^2) H_m(\beta_1 x) H_n(\beta_2 x)\,\mathrm{d}x$$
$$= n!\left(\frac{\pi}{\alpha}\right)^{1/2}\left(-\frac{1-\lambda_1}{1-\lambda_2}\right)^{(m-n)/4} (2(\lambda_1+\lambda_2-1)^{1/2})^{(m+n)/2} p^{|m-n|/2}_{(m+n)/2}$$
$$\times\left[\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2-1}\right)^{1/2}\right] \tag{49}$$

where

$$\lambda_1 = \beta_1^2/\alpha,\ \lambda_2 = \beta_2^2/\alpha.$$

The numbers $m$ and $n$ in the above formulae are of the same parity.
We obtain finally

$$T_n^m = \left(\frac{n!}{m!\xi_\infty}\right)^{1/2} \exp\left[\frac{\mathrm{i}}{2}(m-n)\varphi_{\eta_\infty} - \frac{\mathrm{i}}{2}(m+n)\varphi_{\xi_\infty}\right] p^{(m-n)/2}_{(m+n)/2}\left(\frac{1}{|\xi_\infty|}\right),$$
$$\eta_\infty = |\eta_\infty| \exp(\mathrm{i}\varphi_{\eta_\infty}), \qquad \xi_\infty = |\xi_\infty| \exp(\mathrm{i}\varphi_{\xi_\infty}). \tag{50}$$

The square modulus of the amplitude $T_n^m$ is known to define the probability of a transition between the respective energy states $W_n^m$. It is of interest to consider two special cases of this probability: $\varkappa \to \infty$ and $\varkappa \to 0$.

Restricting ourselves to the second-order terms along the expansion parameter, we obtain the required formulae for $W_n^m$ at $\varkappa \to \infty$ and $\varkappa \to 0$, from which we determine $\varkappa$ in the two extreme cases:

$$\lambda = \left(\frac{2}{A\ddot{y}}\,\frac{W_n^m(\varkappa) - W_n^m(\lambda = 0)}{W_n^m(\lambda = 0)}\right)^{1/2}. \tag{51}$$

Here

$$\lambda = \varkappa^{-1}, \qquad y = \xi_0^{-1}, \qquad \ddot{y} = \tfrac{1}{2}y(w_f - w_i)(7w_f + w_i), \qquad \text{if } \varkappa \to \infty;$$
$$\lambda = \varkappa, \qquad y = (w_f/w_i)^{1/2}\xi_0^{-1}, \qquad \ddot{y} = (y/8w_i^4)(w_i - w_f)(3w_f - w_i), \qquad \text{if } \varkappa \to 0.$$

Besides,

$$W_n^m(\lambda = 0) = \frac{n!}{m!}\, y |p^{(m-n)/2}_{(m+n)/2}(y)|^2, \tag{52}$$

$$A = y^{-1} + 2(1-y^2)^{-1}\left\{\left(1 + \frac{m+n}{2}\right) y - (n+1) p^{(m-n)/2}_{(m+n)/2+1}(y) [p^{(m-n)/2}_{(m+n)/2}(y)]^{-1}\right\}. \tag{53}$$

In the case of $m = n = 0$, one can obtain a simpler formula as compared to (51):

$$\lambda = [(2/y)(W_0^0 - y)]^{1/2}. \tag{54}$$

From this formula it follows that the zero term of the expansion obtained for $W_0^0(\varkappa \to \infty)$ coincides with the square modulus of an overlap integral, i.e. with

$|\int \psi_{0i}^*(x)\psi_{0f}(x)\,dx|^2$ as might be expected. In the latter expression, $\psi_{0i}$ and $\psi_{0f}$ are wavefunctions of the fundamental state of the oscillator (Landau and Lifshitz 1963) with frequencies $w_i$ and $w_f$ respectively.

## 7. Coherent states and Green's functions for a charged particle in a non-stationary magnetic field

The coherent states of a charged particle moving in a non-stationary magnetic field are defined as eigenfunctions of the time-dependent invariants $\hat{A}(t)$ and $\hat{B}(t)$ (see (12) and (13)):

$$\hat{A}|\alpha, \beta, t\rangle = \alpha|\alpha, \beta, t\rangle, \qquad \hat{B}|\alpha, \beta, t\rangle = \beta|\alpha, \beta, t\rangle, \tag{55}$$

where $\alpha$ and $\beta$ are arbitrary complex numbers.

It is not difficult to see that equalities (55) are satisfied by a function having the following form:

$$\begin{aligned}|\alpha, \beta, t\rangle = (m/\pi\hbar\epsilon^2)^{1/2} \exp\{&(\mathrm{i}\dot{\epsilon}m/2\hbar\epsilon)(x^2+y^2) - \tfrac{1}{2}(|\alpha|^2+|\beta|^2) - \mathrm{i}\alpha\beta\epsilon^*/\epsilon \\ &+ (2m/\hbar\epsilon^2)^{1/2}[x(\mathrm{i}\alpha f^* + \beta f) + y(\mathrm{i}\beta f + \alpha f^*)]\}.\end{aligned} \tag{56}$$

The function $|\alpha, \beta, t\rangle$ satisfies the Schrödinger equation with Hamiltonian (11). The coherent states introduced are normalised and complete, but not orthogonal.

Coherent states (56) represent a generating function for $|n_1, n_2, t\rangle$ which are eigenfunctions of the time-dependent invariants $\hat{K}$ and $\hat{L}_z$ (§ 3):

$$|\alpha, \beta, t\rangle = \exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)] \sum_{n_1,n_2=0}^{\infty} \frac{\alpha^{n_1}\beta^{n_2}}{(n_1!\,n_2!)^{1/2}} |n_1, n_2, t\rangle, \tag{57}$$

$$\hat{K}|n_1, n_2, t\rangle = (n_1+\tfrac{1}{2})|n_1, n_2, t\rangle, \qquad \hat{L}_z|n_1, n_2, t\rangle = (n_2-n_1)|n_1, n_2, t\rangle. \tag{58}$$

The explicit form of orthonormalised eigenfunctions $|n_1, n_2, t\rangle$ can be obtained from expression (57) for $|\alpha, \beta, t\rangle$ if one uses the generating function for associated Laguerre polynomials (Lee 1967):

$$\begin{aligned}\exp(xy + ax - a^*y) = \sum_{n_1,n_2=0}^{\infty} \left(\frac{p_1!}{(p+|n_1-n_2|)!}\right)^{1/2} a^{n_2-p} a^{*(p-n_1)}, \\ L_p^{|n_1-n_2|}(|a|^2) x^{n_1} y^{n_2}/(n_1!\,n_2!)^{1/2},\end{aligned} \tag{59}$$

where

$$p = \tfrac{1}{2}(n_1+n_2-|n_1-n_2|).$$

Then

$$\begin{aligned}|n_1, n_2, t\rangle = (-\mathrm{i})^{n_1} &\left(\frac{mp!}{\pi\hbar\epsilon^2(p+|n_1-n_2|)!}\right)^{1/2} \left(\frac{\epsilon^*}{\epsilon}\right)^{(n_1+n_2)/2} \left[\left(\frac{2m}{\hbar|\epsilon|^2}\right)^{1/2} f(x+\mathrm{i}y)\right]^{n_2-p} \\ &\times \left[-\left(\frac{2m}{\hbar|\epsilon|^2}\right)^{1/2} f^*(x-\mathrm{i}y)\right]^{n_1-p} L_p^{|n_1-n_2|}\left(\frac{m}{\hbar}\frac{x^2+y^2}{|\epsilon|^2}\right) \exp\left(\frac{\mathrm{i}\dot{\epsilon}m}{2\hbar\epsilon}(x^2+y^2)\right).\end{aligned} \tag{60}$$

Using the explicit form of the coherent states $|\alpha, \beta, t\rangle$ (56), we can define the Green function of the Schrodinger equation for a charged particle moving in a time-dependent magnetic field:

$$G(x_2, y_2, t_2; x_1, y_1, t_1)$$

$$= \frac{1}{\pi^2} \int |\alpha, \beta, t_2\rangle\langle\alpha, \beta, t_1| \mathrm{d}^2\alpha\, \mathrm{d}^2\beta$$

$$= m(2\mathrm{i}\pi\hbar \sin\gamma|\epsilon_1\epsilon_2|)^{-1} \exp\left\{ -\frac{\mathrm{i}}{4}|R_1|^2 \frac{\mathrm{d}|\epsilon_1|^2}{\mathrm{d}t_1} + \frac{\mathrm{i}}{4}|R_2|^2 \right.$$

$$\left. \times \frac{\mathrm{d}|\epsilon_2|^2}{\mathrm{d}t_2} + \frac{\mathrm{i}}{2}\cot\gamma(|R_1|^2 + |R_2|^2) + \frac{1}{2\mathrm{i}\sin\gamma}(R_1R_2^* + R_1^*R_2) \right\} \tag{61}$$

where $R_i = (2m/\hbar)^{1/2}(x_i + \mathrm{i}y_i)/(|\epsilon_i|)f_i$, $i = 1, 2$ and $z$, $\gamma$ have the same meaning as in (42).

The operators $\hat{X}_0$, $\hat{Y}_0$, $\hat{P}_{0x}$ and $\hat{P}_{0y}$ presented in § 3 are related to the initial conditions of the motion, since on the one hand they are motion integrals and on the other hand, as seen from (16) and (18), at $t = 0$:

$$\hat{X}_0 = (mw_i)^{1/2}(y - \hat{y}_0), \qquad \hat{Y}_0 = (mw_i)^{1/2}\hat{x}_0,$$

$$\hat{P}_{0x} = (mw_i)^{1/2}(x - \hat{x}_0), \qquad \hat{P}_{0y} = (mw_i)^{1/2}\hat{y}_0, \tag{62}$$

where $\hat{x}_0 = (x + \hat{p}_y/mw_i)/2$, $\hat{y}_0 = (y - \hat{p}_x/mw_i)/2$ are operators of orbit-centre coordinates, whereas $(y - \hat{y}_0)$ and $(x - \hat{x}_0)$ are operators of the relative coordinates of the particle under study, moving in a constant magnetic field $H_i$. From (55) and the normalisation condition of the functions $|\alpha, \beta, t\rangle$ it transpires that the mean values of the above operators amount to

$$\langle\hat{X}_0\rangle = \alpha_1/2\hbar, \qquad \langle\hat{P}_{0x}\rangle = \alpha_2/2\hbar,$$

$$\langle\hat{Y}_0\rangle = \beta_1/2\hbar, \qquad \langle\hat{P}_{0y}\rangle = \beta_2/2\hbar, \tag{63}$$

where subscript '1' denotes the real parts, and subscript '2' stands for the imaginary parts of the respective values.

If one considers the classical motion of a charged particle in a stationary uniform magnetic field (equal to $H_i$ in this case), with the coordinates of the centre of the circular orbit equal to $\beta_1/2\hbar$ and $\beta_2/2\hbar$, and the relative coordinates equal to $\alpha_1/2\hbar$ and $\alpha_2/2\hbar$, the quantum motion of the same particle in a coherent state is extremely close to the above classical motion in the sense that the mean values of $\langle\hat{Y}_0\rangle$ and $\langle\hat{P}_{0y}\rangle$ are similar to the coordinates of the centre of the circular orbit of the classical motion, whereas the mean values of $\langle\hat{X}_0\rangle$ and $\langle\hat{P}_{0x}\rangle$ are similar to the relative coordinates of the same motion.

## 8. Probability of transitions for a charged particle in a non-stationary magnetic field

Earlier (§ 3) we supposed that as $t \to \pm\infty$ the magnetic field assumed the values $H_f$ and $H_i$, respectively. Based on this assumption and reasoning in the same way as in the case of an oscillator (§ 5), we shall obtain formula (43) for the amplitude of a transition from the initial to the final state.

It has been established that in the case of a magnetic field the system under discussion passes from the initial state characterised by $\epsilon_i$ (equation (16)) into a state with $\epsilon_\infty$ (equation (34)), which can be expanded in a complete set of final functions with

$\epsilon_f$ (equation (31)), the frequency being expressed in terms of a magnetic field according to formula (15).

Expressions for the initial motion integrals $\hat{A}_i$, $\hat{B}_i$ are obtained from (12) and (13) by substituting $\epsilon = \epsilon_i$, $f = (w_i/2)^{1/2}\epsilon_i$, whereas for the final motion integrals $\hat{A}_f$, $\hat{B}_f$ they can be obtained from the same formulae at $\epsilon = \epsilon_f$, $f = (\omega_f/2)^{1/2}\epsilon_f$. The invariants $\hat{A}$ and $\hat{B}$ as $t \to \infty$ can be expressed in terms of the final operators $\hat{A}_f$ and $\hat{B}_f$ as follows:

$$\hat{A}_\infty = \xi_\infty \hat{A}_f - \mathrm{i}\eta_\infty \hat{B}_f^+, \qquad \hat{B}_\infty = \xi_\infty \hat{B}_f - \mathrm{i}\eta_\infty \hat{A}_f^+, \tag{64}$$

which is checked directly by means of (12), (13), and (34).

Taking into account the above, it is easy to calculate the amplitude relating the coherent states $|\alpha, \beta, i\rangle$ and $|\gamma, \delta, f\rangle$:

$$\begin{aligned} T_{\alpha\beta}^{\gamma\delta} &= \langle \gamma, \delta, f|\alpha, \beta, t\to\infty\rangle = \xi_\infty^{-1} \exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2)] \\ &\quad \times \exp[\xi_\infty^{-1}(\beta\delta^* + \alpha\gamma^* - \mathrm{i}\gamma^*\delta^*\eta_\infty - \mathrm{i}\alpha\beta\eta_\infty^*)]. \end{aligned} \tag{65}$$

From the definition of coherent states in the form of (57) it follows that the amplitude $T_{\alpha\beta}^{\gamma\delta}$ is a generating function for all the other amplitudes. For example, using the generating function for Jacobi polynomials (Malkin *et al* 1970):

$$\begin{aligned} &\exp(ab\eta^* + bd + ac - cd\eta) \\ &\quad = \sum_{n_1,n_2,m_1,m_2=0}^{\infty} \frac{a^{n_1}b^{n_2}c^{m_1}d^{m_2}}{(n_1!\,n_2!\,m_1!\,m_2!)^{1/2}} \\ &\qquad \times s!(n_2+m_1-s)!(-\eta)^{m_1-s}\eta^{*(n_1-s)}(1-|\eta|^2)^s \\ &\qquad \times p_s^{(|m_1-n_1|,|n_1-n_2|)}[(1-|\eta|^2)/(1+|\eta|^2)] \end{aligned} \tag{66}$$

where $s = \frac{1}{2}(n_1+m_1-|n_1-m_1|)$ and $p_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial, one can obtain from the formula for $T_{\alpha\beta}^{\gamma\delta}$, by means of (57), the amplitude of a transition in an energy representation, i.e.

$$\begin{aligned} T_{n_1n_2}^{m_1m_2} &= \langle m_1, m_2, f|n_1, n_2, t\to\infty\rangle = \frac{1}{\xi}\left[\frac{s!(n_2+m_1-s)!}{(s+|n_2-n_1|)!\,(s+|m_2-n_2|)!}\right]^{1/2} \\ &\quad \times \xi_\infty^{*s}\xi_\infty^{s-n_2-m_1}(-\eta_\infty)^{m_1-s}\eta_\infty^{*(n_1-s)}p_s^{(|m_1-n_1|,|n_1-n_2|)}(1-2|\eta_\infty/\xi_\infty|^2). \end{aligned} \tag{67}$$

Here, it has been taken into account that the moment $L_z$ is a motion integral $(n_2-n_1 = m_2-m_1)$. Formula (67) relates to the case of $L_z = n_2 - n_1 \geqslant 0$. If the values $L_z$ are negative, subscripts $1 \leftrightarrow 2$ should be substituted in the formula (67).

By calculating the probability of a transition on the basis of formula (67) in the two extreme cases $\varkappa \to 0$ and $\varkappa \to \infty$ of interest to us, and restricting ourselves to an expansion term quadratic in the parameter $\lambda$, we obtain an expression differing from the respective expression (51) for an oscillator only by the substitution of $W_n^m \to W_{n_1n_2}^{m_1m_2}$, and in this case

$$W_{n_1n_2}^{m_1m_2}(\lambda=0) = \frac{m_2!\,n_1!}{m_1!\,n_2!}(1-y^2)^{m_1-n_1}y^{2(n_2-n_1+1)}|p_{n_1}^{(m_1-n_1,n_2-n_1)}(2y^2-1)|^2, \tag{68}$$

$$\begin{aligned} A &= 2y(1-y^2)^{-1}(n_1-m_1) + 2(n_2-n_1+1)y^{-1} + 4y(m_2+1) \\ &\quad \times p_{n_1-1}^{(m_1-n_1+1,n_2-n_1+1)}(2y^2-1)[p_{n_1}^{(m_1-n_1,n_2-n_1)}(2y^2-1)]^{-1}. \end{aligned} \tag{69}$$

The expression $W^{m_1m_2}_{n_1n_2}$ obtained here relates to the case of $n_2 - n_1 \geqslant 0$, $m_i \geqslant n_i$. At $n_2 - n_1 \geqslant 0$, $m_i \leqslant n_i$ one should substitute $n_i \leftrightarrow m_i$, whereas the case of $n_2 - n_1 < 0$ is obtained from the previous ones by substituting subscripts $1 \leftrightarrow 2$.

The above discussion can be extended to cover also the case of a non-stationary magnetic field produced by an infinite plane of current (Dodonov *et al* 1972).

From the above it is clear that, on the basis of known transition probability values, one can evaluate $\varkappa^{-1}$, the characteristic time of transition from one Landau level to another.

In the case of an oscillator, $\varkappa^{-1}$ is the characteristic time of a transition from the initial to the final stationary state.

It is known that in the Born–Oppenheimer approximation the relative intensity of a vibronic line is given by the square matrix element of the transition moment. In the Condon approximation, however (when ignoring the dependence of the transition moment on the internuclear separation), this relative intensity is defined by the transition probability. In our case it seems to be also reasonable to use the Condon approximation, inasmuch as then we shall be able to evaluate $\varkappa^{-1}$ from experimental data, on the basis of the relative intensity of vibronic lines, using the formula (51).

## Acknowledgment

I wish to express my deep appreciation to Dr V I Man'ko for his statement of the problem and constant attention to this study.

## Appendix

Using equalities (7), (26), and (29), one can obtain new relations for a degenerate hypergeometric function. The substitution of expressions (25) and (28), for $\xi(t)$ and $\eta(t)$, into (29) leads to the relation

$$(1+|d|^2)|\phi(\tfrac{1}{2}, 1+a, b_x)/\phi(\tfrac{1}{2}, 1+a, b)|^2 - |d|^2|\phi(\tfrac{1}{2}-a, 1-a, b_x)/\phi(\tfrac{1}{2}-a, 1-a, b)|^2$$
$$= |\exp[\tfrac{1}{2}(b_x - b)]|^2. \tag{A1}$$

Here and further $|d|^2 = (b/2a)^2(1-b/a)^{-1}$, $b_x = b\exp(-cx)$; $a, b, c$ are positive and finite parameters, $x \geqslant 0$.

Taking into account formulae (25) and (28), and the expression for an arbitrary degenerate hypergeometric function ((9.213) Gradshteyn and Ryzhik 1971), one can transform (26) as follows:

$$(1+|d|^2)^{1/2}\phi(\tfrac{1}{2}-a, 1-a, b)\exp\left(-\frac{acx}{2}\right)\left[\frac{1}{1+a}\phi(\tfrac{3}{2}, 2+a, b_x) - \phi(\tfrac{1}{2}, 1+a, b_x)\right]$$
$$= |d|\phi(\tfrac{1}{2}, 1+a, b)\exp(\tfrac{1}{2}acx).$$
$$\times\left[\frac{1-2a}{1-a}\phi(\tfrac{3}{2}-a, 2-a, b_x) - \phi(\tfrac{1}{2}-a, 1-a, b_x)\right]. \tag{A2}$$

Among the three equalities, namely (7), (26), and (29), only two equalities are independent. That is why the transformation of equality (7) will not give a relation between degenerate hypergeometric functions, different from (A1) and (A2).

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